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# Eigenvector matrices of Cartan matrices for finite groups(Cohomology Theory of Finite Groups and Related Topics)

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CITATION:

Wada, Tomoyuki. Eigenvector matrices of Cartan matrices for finite groups(Cohomology Theory of Finite Groups and Related Topics). 数理解析研究所講究録 2006, 1466: 72-79

ISSUE DATE:

2006-01

URL:

<http://hdl.handle.net/2433/48055>

RIGHT:

## Eigenvector matrices of Cartan matrices for finite groups

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### 1. INTRODUCTION

Let  $G$  be a finite group,  $F$  be an algebraically closed field of characteristic  $p > 0$ , and  $B$  be a block of the group algebra  $FG$  with defect group  $D$ . Let  $C_B = (c_{ij})$  be the Cartan matrix of  $B$  and  $\rho(B)$  be the Frobenius-Perron eigenvalue (i.e. the largest eigenvalue) of  $C_B$ . Let  $(K, R, F)$  be a  $p$ -modular system, where  $R$  is a complete discrete valuation ring of rank one with  $R/(\pi) \simeq F$  for a unique maximal ideal  $(\pi)$  and  $K$  is a quotient field of  $R$  with characteristic 0. Let us denote the number  $l(B)$  of irreducible Brauer characters in  $B$  simply by  $l$ .

We studied on integrality of eigenvalues of the Cartan matrix of a finite group in [4], [17]. Let  $R_B$  and  $E_B$  be the set of all eigenvalues and  $\mathbf{Z}$ -elementary divisors of  $C_B$ , respectively. For cyclic blocks or tame blocks, we proved that  $\rho(B) \in \mathbf{Z}$  if and only if  $R_B = E_B$ , and for any  $p$ -blocks of  $p$ -solvable groups, we proved that  $\rho(B) = |D|$  if and only if  $R_B = E_B$ . Recently, C.C. Xi and D. Xiang proved that a cellular algebra  $A$  is semisimple if and only if all eigenvalues of the Cartan matrix of  $A$  are rational integers and the Cartan determinant is 1 ([15, Theorem 1.1]).

Then, what do eigenvectors of  $C_B$  mean? In this article the author showed that if all eigenvalues of  $C_B$  are rational integers for a cyclic block  $B$ , a tame block  $B$ , a  $p$ -block  $B$  of a  $p$ -solvable group or the principal 3-block  $B$  with elementary abelian Sylow  $p$ -subgroup of order 9, then there exists a unimodular matrix  $U_B$  over  $R$  whose columns consist of eigenvectors of  $C_B$ . We call  $U_B$  an *eigenvector matrix* of  $C_B$ . From Linear Algebra  $U_B$  diagonalizes  $C_B$ . In these cases above, we can take as  $U_B$  actually the Brauer character table matrix for some blocks. For details see [19].

## 2. PRELIMINARIES

We had the following basic conjecture in [4].

Conjecture (Questions 1 and 2 in [4]). *Let  $G$  be a finite group. Let  $B$  be a block of  $FG$  with defect group  $D$ . Then the following are equivalent.*

- (a)  $\rho(B) \in \mathbf{Z}$ .
- (b)  $\rho(B) = |D|$ .
- (c)  $R_B = E_B$ .

For several groups or blocks we proved that Conjecture is true, but we do not yet prove for any finite groups. If this conjecture is true, these conditions must be equivalent to

- (d) *all eigenvalues of  $C_B$  are rational integers.*

Because, (d) implies (a), and (c) implies (d). Here we try to consider proving (d)  $\rightarrow$  (c) in the following section. To begin with, we state some preliminary results in [4]. We first introduce some notation. Let  $\text{IBr}(B) = \{\varphi_1, \dots, \varphi_l\}$  be the set of irreducible Brauer characters in a block  $B$  of  $FG$ . Let  $\{x_1, \dots, x_l\}$  be a set of representatives of  $p$ -regular classes of  $G$  associated with  $B$  ([10, Theorem 11.6]). Let us set  $\varphi_j = {}^t(\varphi_1(x_j), \dots, \varphi_l(x_j))$  for  $1 \leq j \leq l$  and let  $\Phi_B = (\varphi_1, \dots, \varphi_l) = (\varphi_i(x_j))$  be the Brauer character table of  $B$ . Here for a matrix  $A$  we denote by  ${}^tA$  the transposed matrix of  $A$ .

Theorem 1 (Proposition 2 in [4], see also [1, Lemma 4.26 (Lusztig)]). *Let  $B$  be a block of  $FG$  with defect group  $D$ . Suppose  $D \triangleleft G$ . Then the following hold.*

- (1)  $C_B \mathbf{f} = |D| \mathbf{f}$ , where  $\mathbf{f} = {}^t(\varphi_1(1), \dots, \varphi_l(1))$  for  $\{\varphi_1, \dots, \varphi_l\} = \text{IBr}(B)$ .
- (2)  $R_B = E_B = \{|C_D(x_1)|, \dots, |C_D(x_l)|\}$ , where  $\{x_1, \dots, x_l\}$  is a set of representatives of  $p$ -regular classes of  $G$  associated with  $B$ . In particular,

$$C_B \Phi_B = \Phi_B \text{diag}\{|C_D(x_1)|, \dots, |C_D(x_l)|\}.$$

Theorem 2 (Theorem 1 in [4]). *Let  $G$  be a  $p$ -solvable group and let  $B$  be a block of  $FG$  with defect group  $D$ . Then the following are equivalent.*

- (a)  $\rho(B) = |D|$ .

- (b)  $R_B = E_B$ .
- (c) the height of  $\varphi = 0$  for any  $\varphi \in \text{IBr}(B)$ .
- (d)  $\mathbf{f} = {}^t(\varphi_1(1), \dots, \varphi_l(1))$  is an eigenvector for  $\rho(B)$ .

Theorem 3 (Proposition 3 in [4]). *Let  $B$  be a block of  $FG$  with a cyclic defect group  $D$ . Then the following are equivalent.*

- (a)  $\rho(B) \in \mathbf{Z}$ .
- (b)  $\rho(B) = |D|$ .
- (c)  $R_B = E_B$ .
- (d) The Brauer tree  $\Gamma_B$  of  $B$  is a star and its exceptional vertex with multiplicity  $m$ , if it exists, is at the center. In this case

$$C_B = \begin{pmatrix} m+1 & m & \cdots & m \\ m & m+1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m \\ m & \cdots & m & m+1 \end{pmatrix}.$$

- (e)  $B$  is Morita equivalent to its Brauer correspondent block  $b$  of  $FN_G(D)$ .

Theorem 4 (Proposition 4 in [4]). *Let  $B$  be a tame block (not finite type) of  $FG$  with defect group  $D$  (i.e.  $p = 2$  and  $D$  is isomorphic to a dihedral, a generalized quaternion or a semidihedral group). Then the following are equivalent.*

- (a)  $\rho(B) \in \mathbf{Z}$ .
- (b)  $\rho(B) = |D|$ .
- (c)  $R_B = E_B$ .
- (d) One of the following holds.
  - (i)  $l = 1$ ,
  - (ii)  $l = 3, D \simeq E_4$  (an elementary abelian group of order four) and

$$C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

- (iii)  $l = 3, D \simeq Q_8$  (a quaternion group of order eight) and

$$C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

- (e)  $B$  is Morita equivalent to its Brauer correspondent block  $b$  of  $FN_G(D)$ .

### 3. ALL EIGENVALUES OF $C_B$ ARE INTEGERS

Lemma 1 ([5, Proposition 4.5]). *Let  $B$  be a block of  $FG$  with defect group  $D$ . Let  $\lambda$  be an eigenvalue of  $C_B$ . Then there is an algebraic integer  $\mu$  such that  $\lambda\mu = |D|$ . In particular, if  $\lambda$  is a rational integer, then  $\lambda$  is a power of  $p$  dividing  $|D|$ .*

From Linear Algebra there exists a non singular matrix  $U_B$  over the field  $\mathbf{R}$  of real numbers whose column vectors consist of linearly independent  $l$  eigenvectors of  $C_B$  such that  $U_B^{-1}C_B U_B = \text{diag}\{\rho_1, \dots, \rho_l\}$  since  $C_B$  is a real symmetric matrix. We assume that all eigenvalues  $\rho_1, \dots, \rho_l$  of  $C_B$  are rational integers. Then  $\rho_i$  is a power of  $p$  for  $1 \leq i \leq l$  by Lemma 1. We note that in this case we can have an eigenvector  $\mathbf{u}_i$  of  $\rho_i$  being in  $\mathbf{Z}^l$ . Suppose further that  $U_B = (\mathbf{u}_1, \dots, \mathbf{u}_l)$  can be taken as a unimodular matrix over the complete discrete valuation ring  $R$  (i.e.  $U_B \in \text{GL}(l, R)$ ). Then since  $\rho_1, \dots, \rho_l$  are powers of  $p$ , they are also  $\mathbf{Z}$ -elementary divisors of  $C_B$  because  $U_B^{-1}C_B U_B = \text{diag}\{\rho_1, \dots, \rho_l\}$  and  $U_B$  is unimodular. Thus  $R_B = E_B$ . So the following question naturally arises.

Question 1. Let  $G$  be a finite group and  $B$  be a block of  $FG$ . Let  $C_B$  be the Catan matrix of  $B$ . Then can we take a unimodular eigenvector matrix  $U_B$  of  $C_B$  over  $R$ ?

At least does the following hold?

Question 2. Furthermore suppose that all eigenvalues  $\rho_1, \dots, \rho_l$  of  $C_B$  are rational integers. Then can we take a unimodular eigenvector matrix  $U_B$  of  $C_B$  over  $R$ ? i.e. Does there exist  $U_B \in \text{Mat}_l(\mathbf{Z})$  such that  $\det U_B \not\equiv 0 \pmod{p}$ ?

We note that there exists a negative example for Question 2 in a general finite dimensional algebra which is not a finite group algebra.

Example ([16]). Let  $B$  be a Brauer tree algebra. Let  $\Gamma_B$  be the Brauer tree  $\bullet - \circ - \bullet$  with three vertices, where  $\bullet$  means an exceptional vertex with multiplicity  $m$ . Then we have

$$C_B = \begin{pmatrix} m+1 & 1 \\ 1 & m+1 \end{pmatrix}. \text{ So } R_B = \{m+2, m\}, \quad E_B = \{m^2 + 2m, 1\}.$$

Thus eigenvalues of  $C_B$  are rational integers, but  $R_B \neq E_B$  if  $m > 1$ . Actually, we can take an eigenvector  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  for  $m+2$  and an eigenvector  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  for  $m$ . So we can take a eigenvector matrix  $U_B = \begin{pmatrix} \alpha & -\beta \\ \alpha & \beta \end{pmatrix}$  over  $R$  of  $C_B$  for  $\alpha, \beta \in R$ , then  $\det U_B = 2\alpha\beta$ . Therefore, if  $p = 2$ ,  $\det U_B \equiv 0 \pmod{(\pi)}$ . Thus, if  $p = 2$ , we can never take a unimodular eigenvector matrix of  $C_B$  over  $R$ .

If the above  $C_B$  appears as the Cartan matrix of a 2-block of a finite group,  $\det C_B$  must be a power of 2. So  $m = 2$  and  $C_B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ . However, the following results show that this matrix cannot be the Cartan matrix for cyclic blocks, tame blocks,  $p$ -blocks of  $p$ -solvable groups, at least.

#### 4. THEOREMS

We have the following results on Question 2.

**Theorem A.** *Let  $B$  be a cyclic block or a tame block. If  $\rho(B) \in \mathbf{Z}$ , then we can take a unimodular eigenvector matrix  $U_B$  of  $C_B$  over  $R$ . Indeed we have  $U_B = \Phi_b$ , where  $b$  is the Brauer correspondent block of  $B$ .*

**Theorem B.** *Let  $G$  be a  $p$ -solvable group. If  $\rho(B) = |D|$ , then we can take a unimodular eigenvector matrix  $U_B$  of  $C_B$  over  $R$ . Indeed we have  $U_B = \Phi_\beta$  for some block  $\beta$  of a subgroup of  $G$  or a factor group of a central extension of  $G$ .*

*Proof of Theorem A.* Let  $B$  be a cyclic block or a tame block of  $G$ . Then by Theorems 3 and 4 we have that  $B$  and its Brauer correspondent block  $b$  are Morita equivalent. Thus  $C_B = C_b$ . Then we can take  $U_B = \Phi_b$  by Theorem 1, which is unimodular over  $R$ .  $\square$

We use the Fong reduction to prove Theorem B, but we omit it. The following result is due to Koshitani-Kunugi [7] and many author's results (e.g. [8,9,12,13])

through proving Broué's abelian defect group conjecture to be true.

**Theorem C.** *Let  $\tilde{G}$  be a finite group with an elementary abelian Sylow 3-subgroup  $P$  of order 9. Let  $\tilde{B}$  and  $\tilde{b}$  be the principal 3-block of  $\tilde{G}$  and  $N_{\tilde{G}}(P)$ , respectively. Suppose  $O_{3'}(\tilde{G}) = 1$ . Then the following are equivalent.*

- (a)  $\rho(\tilde{B}) \in \mathbf{Z}$ .
- (b)  $\rho(\tilde{B}) = |P| = 9$ .
- (c)  $R_{\tilde{B}} = E_{\tilde{B}}$ .
- (d)  $\tilde{B}$  and  $\tilde{b}$  are Morita equivalent (even stronger Puig equivalent).
- (e) Let  $G := O_{3'}(\tilde{G})$ . Then one of (i) and (ii) holds.
  - (i)  $G = X \times Y$ , where  $X, Y$  are simple groups (abelian or not) with a cyclic Sylow 3-subgroup of order 3, respectively.
  - (ii)  $G$  is one of the following non abelian simple groups with elementary abelian Sylow 3-subgroup of order 9.
    - (1)  $\text{PSU}_3(q^2)$  with  $2 < q \equiv 2$  or  $5 \pmod{9}$ .
    - (2)  $\text{PSp}_4(q)$  with  $q \equiv 4$  or  $7 \pmod{9}$ .
    - (3)  $\text{PSL}_5(q)$  with  $q \equiv 2$  or  $5 \pmod{9}$ .
    - (4)  $\text{PSU}_4(q^2)$  with  $q \equiv 4$  or  $7 \pmod{9}$ .
    - (5)  $\text{PSU}_5(q^2)$  with  $q \equiv 4$  or  $7 \pmod{9}$ .

In these cases, we can take  $\Phi_{\tilde{b}}$  as a unimodular eigenvector matrix  $U_{\tilde{B}}$  of  $C_{\tilde{B}}$ .

*Proof.* (e)  $\rightarrow$  (d). Then [7, (5.3),(5.6)] states that  $\tilde{B}$  and  $\tilde{b}$  are Puig equivalent.

It is easy to see that (d)  $\rightarrow$  (c), because  $C_B = C_b$  and by Theorem 1. It is obvious that (c)  $\rightarrow$  (b) and (b)  $\rightarrow$  (a).

(a)  $\rightarrow$  (e). Suppose (e) does not hold. Then  $G$  is one of the following alternating groups or sporadic simples  $A_6, A_7, A_8, M_{11}, M_{22}, M_{23}, M_{24}, HS$  or one of the following simple groups of Lie type ;  $\text{PSL}_3(q)$  for  $q \equiv 4$  or  $7 \pmod{9}$ ,  $\text{PSp}_4(q)$  for  $2 < q \equiv 2$  or  $5 \pmod{9}$ ,  $\text{PSL}_4(q)$  for  $2 < q \equiv 2$  or  $5 \pmod{9}$ . But for these simple groups we can easily check that  $\rho(B_0(FG))$  is not a rational integer, here  $B_0$  means the principal 3-block. So we can prove that neither is  $\rho(B_0(F\tilde{G}))$  by the following Proposition.

**Proposition 3.** *Let  $G$  be a finite group and  $H \triangleleft G$  with  $|G : H| = q$ , where  $q$  is a prime number distinct from  $p$ . Let  $b$  be a block of  $FH$  and  $B_1, \dots, B_m$  be all blocks of  $G$  covering  $b$ . Then  $\rho(B_i) = \rho(b)$  for all  $1 \leq i \leq m$ .*

We skip to prove Proposition 3. Here for example we show a typical case. Other cases are similar.

$$\text{If } G = M_{11}, \text{ then } C = \begin{pmatrix} 5 & 2 & 2 & 3 & 3 & 0 & 1 \\ 2 & 3 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 3 & 1 & 2 & 0 & 1 \\ 3 & 2 & 1 & 4 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 & 4 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 1 & 2 \end{pmatrix} \text{ and}$$

$$f_C(x) = (x^2 - 3x + 1)(x^5 - 20x^4 + 102x^3 - 192x^2 + 135x - 27).$$

Since  $10 < \rho(B) < 16$ ,  $\rho(B)$  is not an integer by [5, Lemma 3.1 (2)].

Thus we have proved (a)  $\rightarrow$  (e). Then by Lemma 2 we can take  $U_B = \Phi_b$ . So there exists a unimodular eigenvector matrix of  $C_B$  in this case.  $\square$

Remark 1. There are small misprints in [7, (5.7) Lemma]. In Case 2 and Case 4 the small star marks should be the big star marks.

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